TRACKING EIGENVALUES TO THE FRONTIER OF MODULI SPACE II: LIMITS FOR EIGENVALUE BRANCHES

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1. INTRODUCTION

Let $I \subset \mathbb{R}$ denote a compact interval symmetric about 0 and let $c \cdot I$ denote the dilation of I by a factor of c. Let M and N, be compact oriented smooth manifolds of dimensions d and $d + 1$ respectively. We suppose that we have an embedded copy of $3 \cdot I \times M$ inside of N. (See Figure 1). Let N^0 denote the complement of the hypersurface $\{0\} \times M$.

We consider families $\epsilon \to g(\epsilon)$ of Riemannian metric (tensors) on N each of whose restriction to $3 \cdot I \times M$ is a warped product of the following form

$$(1) \qquad g(\epsilon)|_{3 \cdot I \times M} \;=\; \rho(\epsilon, t)^{2a} \, dt^2 + \rho(\epsilon, t)^{2b} \, h.$$

Here h is a fixed Riemannian metric on M, ρ is smooth positive function that is positively homogeneous of degree 1 on $\mathbb{R}^2 \setminus \{\vec{0}\}$, and a and b are real numbers.

The 'limiting metric' $g(0)$ is singular along the hypersurface $\{0\} \times M$ provided $(a, b) \neq \vec{0}$. Indeed, since ρ is homogeneous

$$g(0)|_{3 \cdot I \times M} \;=\; (c_\pm t)^{2a} \, dt^2 + \; (c_\pm t)^{2b} \, h.$$

for some homogeneity constants $c_\pm$. Melrose [Mlr] has observed that the metric $g(0)$ is Riemannian complete if and only if $a \leq -1$, whereas $g(0)$ has finite volume if and only if $a + bd > -1$. (See Figure 2).

Example 1.1 (Hyperbolic Degeneration)**.** Let γ be a simple closed curve in a compact oriented surface N with $\chi(N) < 0$. Let g_ϵ be a metric on N of constant curvature -1 such that the unique geodesic homotopic to γ has length $\epsilon < 2 \cosh^{-1}(2)$. By the collar lemma, there exists an embedding $I \times \gamma \to N$ with $I = [-\frac{1}{3}, \frac{1}{3}]$ such that

$$(2) \qquad g_\epsilon|_{3 \cdot I \times \gamma} \;=\; \frac{dt^2}{\epsilon^2 + t^2} + (\epsilon^2 + t^2) \, dx^2$$

where x is the usual coordinate on the circle $\mathbb{R}/\mathbb{Z} \cong \gamma$. Note that the Riemannian surface $(I \times \gamma)^0, g_0)$ is a union of hyperbolic cusps. In this special case, $\rho(\epsilon, t) = (\epsilon^2 + t^2)^{\frac{1}{2}}$, $a = -1$, and $b = 1$.

Henceforth, we will assume that ρ is strictly convex along nonradial lines and that $\partial_\epsilon \rho \geq 0$ for $\epsilon \geq 0$. Moreover, we assume that the restriction of $g'(\epsilon)/g(\epsilon)$ to the unit tangent bundle of $K = N \setminus (I \times M)$ is bounded.[1] With these assumptions we have

Theorem 1.2 (Main Theorem). *Let $b > 0$, and either let $a < -1$ or let $a = -1$ and $a + bd = 0$. Suppose that $\epsilon \to g(\epsilon)$ is a real-analytic family of Riemannian metrics on N satisfying (1). Then each eigenvalue branch of the associated family of Laplacians, $\Delta_{g(\epsilon)}$, converges to a finite limit as ϵ tends to 0^+.*

S. Wolpert [Wlp92] proved Theorem 1.2 in the special case of hyperbolic degeneration. He subsequently used this convergence to produce evidence supporting the belief that Maass cusp forms 'disappear' under perturbation [Wlp94] [Snk95]. Note that although the present paper does not include a discussion of manifolds with cusps, the methods described here apply equally well to the eigenbranches of a 'pseudo-Laplacian' associated to a manifold N with finitely many cusps.

By combining results of this paper with those of the prequel [Jdg01], we obtain

Theorem 1.3. *Let $a \leq -1$ and $b > 0$. Let $\psi(\epsilon)$ be an eigenfunction branch whose zeroth Fourier coefficient (see §3) vanishes identically for small ϵ. Then $\psi(\epsilon)$ converges to an $L^2(N^0, g(0))$-eigenfunction of $\Delta_{g(0)}$.*

Since real-analytic eigenbranches can 'cross', their tracking is far subtler than the continuity of ordered eigenvalues. For example, consider the family of Laplacians, Δ_ϵ, associated to the flat tori $\mathbb{R}^2/(\epsilon\mathbb{Z} \bigoplus \epsilon^{-1}\mathbb{Z})$. In this case, almost all of the real-analytic eigenvalue branches tend to infinity as ϵ tends to zero. Yet, there are infinitely many branches that tend to zero. Therefore if for each $\epsilon > 0$, one were to label the eigenvalues in increasing order (with multiplicities)

$$(3) \qquad 0 < \lambda_1(\epsilon) \leq \lambda_2(\epsilon) \leq \cdots \leq \lambda_k(\epsilon) \leq \cdots,$$

then each $\lambda_k(\epsilon)$ would tend to zero as ϵ tended to zero.[2]

Therefore, although (1) describes a relatively narrow class of geometric degenerations, the conclusion of Theorem 1.2 provides a great deal more information concerning spectral behavior than the usual convergence results concerning ordered eigenvalues. (See, for example, the recent work of Cheeger and Colding [ChgCld99].) Indeed, the geometer's standard tool for estimating the size of eigenvalues—the minimax principle—cannot be used to track real-analytic eigenvalue branches due to possible eigenbranch 'crossings'.

Here, we rely instead on the variational principle $\dot{\lambda} = \int \psi \dot{\Delta} \psi$. To illustrate our use of this principle, we prove in §2 the following general result:

Theorem 1.4. *Let $g(\epsilon)$ be a real-analytic family of metrics on a Riemannian manifold N. Then for each real-analytic eigenbranch we have*

$$(4) \qquad \left| \frac{\lambda'(\epsilon)}{\lambda(\epsilon)} \right| \leq (\dim(N) + 1) \left\| \frac{g'(\epsilon)}{g(\epsilon)} \right\|.$$

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[1] Here we view g and g' as quadratic forms on each tangent space; thus each may be regarded as function on the unit tangent bundle.

[2] Note that although $\lambda_k(\epsilon)$ is continuous, it is not real-analytic.
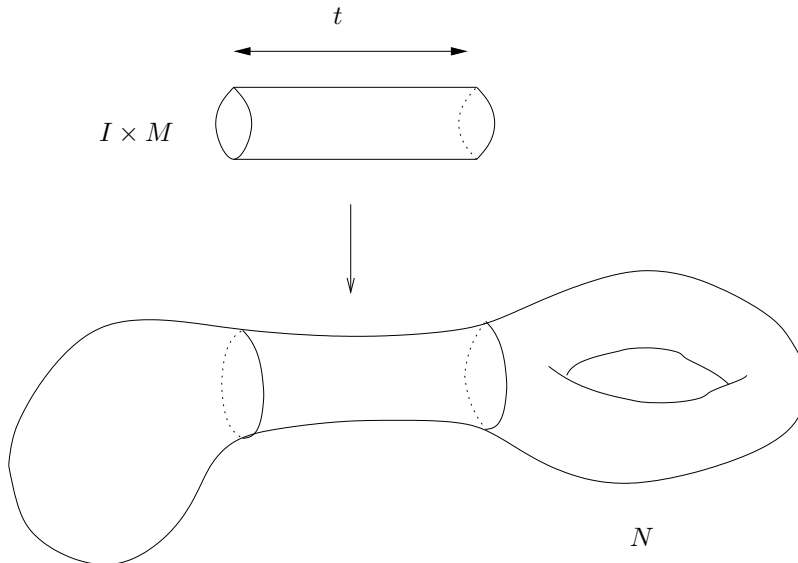
FIGURE 1. The set up
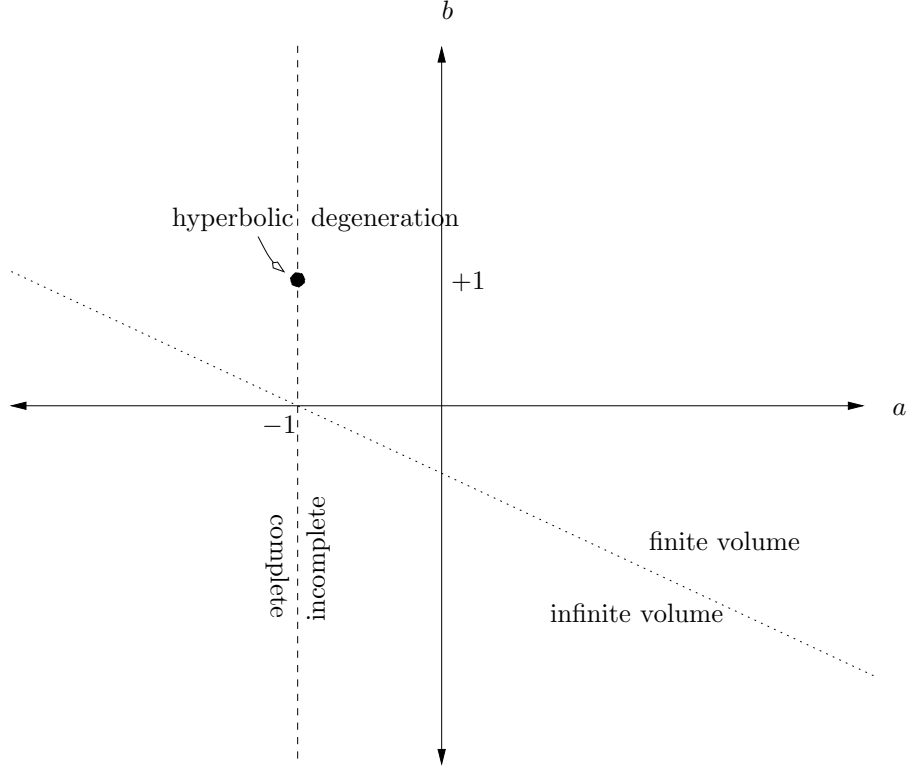
For a family $g(\epsilon)$ satisfying (1), there are vector fields supported in $I \times M$ such that the right hand side of (4) is unbounded as ϵ tends to zero. Thus, in order to use the variational principle to prove Theorem 1.2, one must exhibit some control over the size of eigenfunctions in the bicollar $I \times M$. For large eigenvalues, controlling the size of eigenfunctions is notoriously difficult [Snk95] [Zld00]. Indeed, for $b > 0$, the central hypersurface $\{0\} \times M$ is totally geodesic, and hence the *correspondence principle* of quantum physics leads one to 'expect'—perhaps erroneously—that the mass of an eigenfunction with large eigenvalue concentrates near $\{0\} \times M$. The possibility of such 'scarring' on $\{0\} \times M$ greatly contributes to the delicacy of the proof of Theorem 1.2. Fortunately, the ill-effects of possible 'scarring' are ameliorated by the inequality $a \leq -1$, that is, by the completeness of the limiting manifold.

We devote the remainder of this paper, with the exception of §5, to proving Theorem 1.2. We now outline the contents and hence also the proof. In §2 we illustrate our use of the variational principle with a proof of Theorem 1.4. In §3, we establish some basic facts concerning the warped product (1) including an integration by parts formula (Lemma 3.4) on which most of our analysis is based.

Underlying the proof of Theorem 1.2 is a basic fact: A nonnegative function $f \in C^1(\mathbb{R}^+)$ has a finite limit as ϵ tends to 0^+ provided the negative variation of f over $]0, \epsilon]$ tends to zero as ϵ tends to zero. Towards applying this to an eigenbranch λ, we derive in §4 lower bounds for the derivative λ'. As an example of our approach, we use these lower bounds in §5 to prove

Theorem 1.5. *For $a < -1$ and $b \leq 0$, each eigenvalue branch converges to a finite limit as ϵ tends to 0^+. For $a = -1$ and $b < 0$, each eigenvalue branch remains bounded as ϵ tends to 0^+.*

(Future work will include a more thorough investigation of the cases in Theorem 1.5 as well as a study of the 'adiabatic' case $(a, b) = (-1, 0)$.)

FIGURE 2. The (a, b) parameter space

Beginning with §6, we restrict attention to the case of interest in the present work: $a \leq -1$ and $b < 0$. We show in §6 that $\epsilon^{2b} \cdot \lambda(\epsilon)$ converges to a finite limit (Theorem 6.2). In §7, we find that if $\epsilon^{2kb} \cdot \lambda(\epsilon)$ remains bounded for some $k < 1$, then $\lambda(\epsilon)$ converges to a finite limit (Theorem 6.2). In §8 boundedness for $k < 1$ is verified provided $\mu^* = \lim \rho^{2b}(\epsilon, 0) \cdot \lambda(\epsilon)$ is not a positive eigenvalue of the Laplacian Δ_h for (M, h). Hence in this case the eigenbranch has a finite limit (Theorem 8.1). In §9 we assume that μ^* is a positive eigenvalue of Δ_h, and obtain a contradiction in the form of two conflicting estimates: Lemmas 9.2 and 9.3. Theorem 1.2 follows.

We remark that the condition $\mu^* \in Spec(\Delta_h) \setminus \{0\}$—and hence the threshold $k < 1$—is intimately tied to 'scarring'. Indeed, one finds that the projection of ψ onto the μ^*-eigenspace is a scarring mode in the sense of, for example, §7 of [CdVPrs94]. For the purpose of proving Lemma 9.2 we need only know that the 'width' of a scar is $O(\lambda^{-\frac{1}{4}})$ as λ tends to infinity. This result is given in Appendix A.

The reader familiar with §3 in [Wlp92] will recognize the thread of the argument outlined above. Indeed, not only does the case of hyperbolic degeneration serve as motivation for the present work, many of its basic features are representative of the general case. On the other hand, at this level of generality, we cannot avail ourselves of Teichmüller theory nor the Poincaré series estimate of [Wlp92]. Moreover, the peculiar features of the 'overcomplete' case $a < -1$ do not appear in hyperbolic

degeneration. These features add complication to the arguments, especially to those found in §9.

I thank my ever-patient wife, Nacy, for her support. I also thank the referee for generous help wih the exposition.

2. Eigenvalue variation in the space of metrics

Let $\mathcal{M}(N) \subset \otimes^2 TN^*$ be the space of all (smooth) Riemannian inner products on a compact manifold N. To each $g \in \mathcal{M}(N)$ we associate the Laplacian Δ_g. This is a self-adjoint, unbounded operator on $L^2(N, dV_g)$ defined via the Friedrich's extension with respect to symmetric boundary conditions.

A fixed inner product $g^* \in \mathcal{M}(N)$ induces a Banach norm on the space of 2-tensors $\otimes^2 TN^*$. A family of metric tensors $\epsilon \to g(\epsilon)$ is said to be real-analytic if it defines a real-analytic path in the Banach space $\otimes^2 TN^*$. Using the ratio of Riemannian measures dV_g/dV_{g^*}, one constructs a natural family of unitary operators that conjugates $\epsilon \to \Delta_{g(\epsilon)}$ into a real-analytic family of compactly resolved operators that are self-adjoint with respect to the fixed sesquilinear form determined by dV_{g*}. It follows from analytic perturbation theory [Kat] that there exists a countable collection of eigenfunction branches, $\{\epsilon \to \psi_k(\epsilon)\} \subset L^2(N, dV_g) \cap C^\infty(N)$, such that for each fixed ϵ, the set $\{\psi_k(\epsilon)\}$ is an orthonomal basis for $L^2(N, dV_g)$.

Given a continuous function $f : TN^* \setminus \{0\} \to \mathbb{R}$ satisfying $f(c \cdot v) = f(v)$ for all $c \in \mathbb{R}$, let $||f||$ denote the supremum. An example of such a function is $v \to h(v,v)/g(v,v)$ where $g \in \mathcal{M}(N)$ and $h \in \otimes^2 TN^*$ is an arbitrary 2-tensor.

Theorem 2.1. *Let $g(\epsilon)$ be a real-analytic family of metrics on N. Then for each real-analytic eigenbranch we have*

$$(5) \qquad \left| \frac{\lambda'(\epsilon)}{\lambda(\epsilon)} \right| \leq (\dim(N) + 1) \left\| \frac{g'(\epsilon)}{g(\epsilon)} \right\|.$$

Proof. We fix a background metric g^* and write dV^* for its volume form. Define $\alpha \in C^\infty(N)$ by $dV_g(\epsilon) = \alpha(\epsilon) \cdot dV^*$. Let $\psi(\epsilon)$ be an eigenfunction branch corresponding to $\lambda(\epsilon)$. Supressing subscripts, we have

$$(6) \qquad \int g(\nabla\psi, \nabla\psi)\alpha \, dV^* = \lambda \int \psi^2 \alpha \, dV^*.$$

Each object in (6) is real-analytic in ϵ. By Taylor expanding, collecting first order terms, integrating by parts, and using the eigenequation, we find that

$$(7)$$
$$\dot{\lambda} \int \psi^2 dV = \int \dot{g}(\nabla\psi, \nabla\psi)dV + 2\int g(\dot{\nabla}\psi, \nabla\psi)dV + \int \frac{\dot{\alpha}}{\alpha}\left(g(\nabla\psi, \nabla\psi) - \lambda\psi^2\right)dV$$

Here the symbol $\cdot$ denotes the first derivative with respect to ϵ evaluated at $\epsilon = 0$.

By definition, we have $g(\nabla f, X) = X \cdot f$ for each fixed vector field X and function f on N. Differentiating in t yields $\dot{g}(\nabla f, X) + g(\dot{\nabla} f, X) = 0$. Using this identity, (7) reduces to

$$(8) \qquad \dot{\lambda} \int \psi^2 dV = -\int \dot{g}(\nabla\psi, \nabla\psi)dV + \int \frac{\dot{\alpha}}{\alpha}\left(g(\nabla\psi, \nabla\psi) - \lambda\psi^2\right)dV.$$

From $|\dot{g}(\nabla\psi, \nabla\psi)/g(\nabla\psi, \nabla\psi)| \leq ||\dot{g}/g||$, we have

$$(9) \qquad \left| \int \dot{g}(\nabla\psi, \nabla\psi) dV \right| \leq ||\dot{g}/g|| \cdot \lambda \cdot \int \psi^2 dV.$$

By interpreting $\alpha(\epsilon)$ as the determinant of the matrix representation of $g(\epsilon)$ with respect to an orthonormal basis of g^*, one finds that the supremum of $|\dot{\alpha}|$ is bounded by $\dim(N) \cdot ||\dot{g}/g||$. The claim follows by applying this bound and (9) to (8). $\qquad\square$

3. Preliminaries concerning the warped product

We record some basic facts concerning the Laplacian, its eigenvalues, and eigen-functions, on $I \times M$ with the metric given in (1). In the following Δ_h, ∇_h, and dV_h, will denote respectively, the Laplacian, gradient, and volume form, associated to the metric h on a fibre $\{t\} \times M$.

Recall that d is the dimension of M. For any $f \in C_0^\infty(I \times M)$

$$(10) \qquad \Delta_g f = -L(f) + \rho^{-2b} \Delta_h f$$

where

$$(11) \qquad L(f) = \rho^{-a-bd} \, \partial_t \, \rho^{-a+bd} \, \partial_t \, f.$$

and

$$(12) \qquad \nabla_g f = \rho^{-2a} \cdot \partial_t f + \rho^{-2b} \cdot \nabla_h f.$$

The volume form restricted to $I \times M$ is

$$(13) \qquad dV_g = \rho^{a+bd} \, dt \, dV_h.$$

Remark 3.1. If no subscript appears, then the object is associated to g.

Given $f : I \times M \to \mathbb{R}$, define

$$(14) \qquad ||f||_M^2(t) = \int_{\{t\}\times M} f^2(t, m) \, dV_h.$$

Proposition 3.2. *Let* $\psi \in C^2(I \times M)$ *satisfy* $\Delta_g \psi = \lambda\psi$. *Then*

$$(15) \qquad \frac{1}{2} L\left(||\psi||_M^2 \right) = -\lambda \cdot ||\psi||_M^2 + \int_{\{t\}\times M} g(\nabla\psi, \nabla\psi) \, dV_h.$$

Proof. Straightforward computation gives

$$\frac{1}{2} L(||\psi||_M^2) = \frac{1}{2} \int_M L(\psi^2) dV_h = \int_M \psi L(\psi) dV_h + \rho^{-2a} \int_M (\partial_t \psi)^2 \, dV_h.$$

From (10) and $\Delta\psi = \lambda\psi$ we find that

$$-\int_M \psi L(\psi) \, dV_h + \rho^{-2b} \int_M \psi \cdot \Delta_h \psi \, dV_h = \lambda \int_M \psi^2 \, dV_h.$$

Integrating by parts over M gives

$$(16) \qquad \int_M \psi \cdot \Delta_h \psi \, dV_h = \int_M h(\nabla_h \psi, \nabla_h \psi) \, dV_h.$$

Also note that from (12) we have

$$(17) \qquad g(\nabla\psi, \nabla\psi) = \rho^{-2a}(\partial_t\psi)^2 + \rho^{-2b}h(\nabla_h\psi, \nabla_h\psi).$$

The claim follows. $\qquad\square$

Corollary 3.3. *Let $\psi \in C^2(I \times M)$ satisfy $\Delta_g\psi = \lambda\psi$. Suppose that for each $t \in I$,*

$$(18) \qquad \int_{\{t\}\times M} (\psi \cdot \Delta_h\psi) \ dV_h \ \geq \ \mu \cdot \int_{\{t\}\times M} \psi^2 \ dV_h.$$

Then

$$(19) \qquad \frac{1}{2}L\left(||\psi||_M^2\right) \ \geq \ \left(\mu\rho^{-2b} - \lambda\right) \cdot ||\psi||_M^2.$$

Proof. Apply Proposition 3.2 and (17). $\qquad\square$

Lemma 3.4 (Integration by Parts Formula). *Let $\sigma \in C^\infty(\mathbb{R}^2\setminus\{0\})$ be positive and positively homogeneous of degree s. There exist constants C, C', such that for any eigenpair (ψ, λ) on $(2 \cdot I) \times M$ we have*

$$\left|\int_{I\times M} \sigma \cdot \left(g(\nabla\psi, \nabla\psi) - \lambda\psi^2\right) \ dV\right| \leq C \int_{I\times M} \rho^k\psi^2 dV \ + \ C'(\lambda+1) \int_{(2I\setminus I)\times M} \psi^2 \ dV$$

where $k = -2a - 2 + s$.

Proof. Let J denote the dilated interval $\sqrt{2} \cdot I$, and let $0 \leq \chi \leq 1$ belong to $C_0^\infty(J)$ with $\chi \equiv 1$ on I. By multiplying both sides of (15) by $\chi\sigma$ and integrating over J one obtains

$$(20) \qquad \int_J \chi\sigma \cdot L(||\psi||_M^2)\rho^{a+bd}dt = \int_{J\times M} \chi\sigma \cdot \left(g(\nabla\psi, \nabla\psi) - \lambda\psi^2\right) \ dV_g.$$

On the other hand, integration by parts gives

$$(21) \qquad \int_J \chi\sigma \cdot L(||\psi||_M^2)\rho^{a+bd}dt = \int_J L(\chi\sigma) \cdot ||\psi||_M^2\rho^{a+bd}dt.$$

We have $L(\chi\sigma) = \chi L(\sigma) + f$ where $f \equiv 0$ in a neighborhood of the origin. Note that L adds $-2a-2$ to the homogeneity of any function. Hence $\deg(L(\sigma)) = -2a-2+s$, and thus, since $\rho > 0$, we have $|L(\sigma)| = O(\rho^k)$. Therefore, by (21)

$$\left|\int_J \chi\sigma \cdot L(||\psi||_M^2)\rho^{a+bd}dt\right| \ \leq \ \int_{J\times M} \rho^k \cdot \psi^2 \ dV \ \leq \ \int_{I\times M} \rho^k\psi^2 dV + C \int_{(2I\setminus I)\times M} \psi^2 dV$$

where $k = -2a - 2 + s$.

To complete the proof it suffices to show that

$$(22) \qquad \left|\int_{J\times M} (1 - \chi) \cdot \sigma \cdot g(\nabla\psi, \nabla\psi) \ dV\right| \leq C'(\lambda+1) \int_{(2I\setminus I)\times M} \psi^2 \ dV.$$

Since $\sigma > 0$ and σ is bounded on the support of $1 - \chi$, there exists C' such that

$$\int_{J\times M} (1 - \chi) \cdot \sigma \cdot g(\nabla\psi, \nabla\psi) \ dV \ \leq \ C' \int_{J\times M} (1 - \chi) \cdot g(\nabla\psi, \nabla\psi) \ dV$$

Let $\eta(t) = (1 - \chi(t)) \cdot \chi(t/\sqrt{2})$. Integrating by parts gives

$$\int_{(2 \cdot I) \times M} \eta \cdot g(\nabla\psi, \nabla\psi) \, dV = \lambda \int_{(2 \cdot I) \times M} \eta \cdot \psi^2 dV + \int_{(2 \cdot I) \times M} \psi \cdot g(\nabla\psi, \nabla\eta) \, dV.$$

Note that the support of $\nabla\eta$ belongs to $(2I \setminus I) \times M$. Integration by parts in t gives

$$\int_{(2I) \times M} \psi \cdot g(\nabla\psi, \nabla\eta) \, dV = \int_{(2I) \times M} \psi^2 \cdot \partial_t^2 \eta \, dV.$$

Estimate (22) then follows from the fact that η has support in $(2 \cdot I) \setminus I$ and equals $1 - \chi$ on J.   □

Remark 3.5. Suppose that $k \leq 0$. Then $\epsilon^{-k} \cdot \rho^k$ is well-defined, continuous, and homogeneous of degree 0. Hence it is bounded. Thus, there exists a constant C such that

$$(23) \qquad\qquad\qquad \rho^k(\epsilon, t) \leq C \cdot \epsilon^k$$

for all $(\epsilon, t) \in \mathbb{R}^2 \setminus \{\vec{0}\}$.

Remark 3.6. Since ρ is homogeneous and strictly convex along nonradial lines, there exists $c \in \mathbb{R}$ such that for each $\epsilon \neq 0$, the function $t \to \rho(\epsilon, t)$ has a unique maximum at $t = c\epsilon$. To prove 1.2, without loss of generality, we may assume that $c = 0$. For otherwise, based on the linear map $t \to t - \epsilon c$, one may construct a real-analytic family of diffeomorphisms $\phi_\epsilon : N \to N$ such that

$$(24) \qquad\qquad \phi_\epsilon^* g(\epsilon)|_{2I \times M} = \rho^{2a}(\epsilon, t - c\epsilon) \, dt^2 + \rho^{2b}(\epsilon, t - c\epsilon) \, h.$$

Then one works with the positive, positively homogeneous function $\rho(\epsilon, t - c\epsilon)$.

Proposition 3.7. *For each $\epsilon \geq 0$, the maximum of the function*

$$(25) \qquad\qquad\qquad \sigma_\epsilon(t) = \frac{\dot{\rho}}{\rho} \, (\epsilon, t)$$

is ϵ^{-1}. This maximum is uniquely achieved at $t = 0$.

Proof. Note that by homogeneity and positivity, $\rho(\epsilon, 0) = c \cdot \epsilon$ for some $c > 0$, and hence $\sigma_\epsilon(0) = \partial_\epsilon \log(\rho)(\epsilon, 0) = \epsilon^{-1}$. Therefore, the first claim will follow from the second.

By Remark 3.6, we may assume that the function $t \to \rho(\epsilon, t)$ has a uniques minimum at $t = 0$, and thus ρ^{-1} has a unique maximum there. To prove the claim, it will suffice to show the same for $\dot{\rho}$. In other words, it is enough to show that $\partial_t \partial_\epsilon \rho(\epsilon, t)$ is positive for $t < 0$ and negative for $t > 0$.

Since $t \to \rho(\epsilon, t)$ is strictly convex, $t \to \partial_t \rho(\epsilon, t)$ is strictly increasing. Let $0 \leq \epsilon_1 \leq \epsilon_2$. The derivative $\partial_t \rho$ is homogeneous of degree 0, and, therefore, for $t > 0$

$$(26) \qquad\qquad \partial_t \rho(\epsilon_2, t) = \partial_t \rho\left(\epsilon_1, \frac{\epsilon_1}{\epsilon_2} t\right) \leq \partial_t \rho(\epsilon_1, t).$$

Hence $\epsilon \to \partial_t \rho(\epsilon, t)$ is decreasing for $\epsilon \geq 0$. That is, $\partial_\epsilon \partial_t \rho(\epsilon, t)$ is negative for $t > 0$ as desired.

An analogous argument shows that $\partial_\epsilon \partial_t \rho(\epsilon, t) > 0$ for $t < 0$. The claim follows. $\square$

4. LOWER BOUNDS FOR DEGENERATING FAMILIES

The purpose of this section is to derive a useful lower bound for $\dot\lambda$. Towards this end, we define the *zeroeth Fourier coefficient*, ψ_0, of a function ψ on $I \times M$ by

$$(27) \qquad \psi_0(t) \;=\; \int_M \psi(t, m) \; dV_h(m)$$

and the *complement*, $\widehat{\psi}$, by

$$(28) \qquad \widehat{\psi}(t, m) \;=\; \psi(t, m) - \psi_0(t).$$

Note that ψ is a Δ_g eigenfunction with eigenvalue λ, if and only if both ψ_0 and $\widehat{\psi}$ are.

In the sequel, K denotes the set complement $N \setminus (I \times M)$.

Theorem 4.1. *Let $a \leq 0$. There exist positive constants C, C' such that for each eigenbranch (ψ, λ)*

$$\dot\lambda \int_N \psi^2 \; dV \;\geq\; -2\max\{a, b\} \cdot \lambda \int_{I \times M} \frac{\dot\rho}{\rho} \cdot \widehat{\psi}^2 \; dV$$

$$(29) \qquad\qquad - C \int_{I \times M} \rho^{-2a-3} \cdot \psi^2 \; dV \;-\; C'(\lambda + 1) \int_K \psi^2 \; dV.$$

Moreover, if $a + bd = 0$, then the integrand $\rho^{-2a-3} \cdot \psi^2$ can be replaced with $\rho^{-2a-3} \cdot \widehat{\psi}^2$.

Proof. Our starting point is formula (8):

$$(30) \quad \dot\lambda \int_N \psi^2 \; dV \;=\; -\int_N \dot g(\nabla\psi, \nabla\psi) \; dV \;+\; \int_N \frac{\dot\alpha}{\alpha} \cdot \big(g(\nabla\psi, \nabla\psi) - \lambda\psi^2\big) \; dV.$$

Recall that by (global) hypothesis, the supremum of $|g'(\epsilon)/g(\epsilon)|$ over the unit tangent bundle of K is finite. By applying the argument that immediately follows (8) to the restriction of $g(\epsilon)$ to TK, we obtain

$$(31) \quad \left| -\int_K \dot g(\nabla\psi, \nabla\psi) dV + \int_K \frac{\dot\alpha}{\alpha} \big(g(\nabla\psi, \nabla\psi) - \lambda\psi^2\big) dV \right| \leq C \cdot \lambda \int_K \psi^2 dV.$$

for some positive constant C. By (13), the restriction of $\dot\alpha/\alpha$ to $I \times M$ equals $(a + bd)\dot\rho/\rho$. Thus, by combining (30) and (31) we obtain

$$\dot\lambda \int_N \psi^2 \; dV \;\geq\; -\int_{I \times M} \dot g(\nabla\psi, \nabla\psi) \; dV$$

$$(32) \qquad\qquad + (a + bd) \int_{I \times M} \frac{\dot\rho}{\rho} \cdot \big(g(\nabla\psi, \nabla\psi) - \lambda\psi^2\big) \; dV$$

$$\qquad\qquad - C \cdot \lambda \int_K \psi^2 \; dV.$$

We claim that

$$\int_{I \times M} \dot{g}(\nabla \psi, \nabla \psi) dV \ \leq \ \int_{I \times M} \dot{g}(\nabla \widehat{\psi}, \nabla \widehat{\psi}) dV. \tag{33}$$

To see this, first note that from (1) we compute

$$\dot{g}|_{I \times M} \ = \ \frac{\dot{\rho}}{\rho} \cdot (2a\rho^{2a} dt^2 + 2b\rho^{2b} h). \tag{34}$$

The function ψ_0 is constant on each fibre $\{t\} \times M$, and hence $h(\nabla \psi_0, \nabla \psi_0) = 0$. Therefore, since $a \leq 0$ and $\dot{\rho} \geq 0$, we find that

$$\dot{g}(\nabla \psi_0, \nabla \psi_0) \ = \ 2a \cdot \dot{\rho} \cdot \rho^{2a-1} \cdot (\partial_t \psi_0)^2 \leq 0 \tag{35}$$

The operator ∂ preseves the decomposition $\widehat{f} + f_0$. In particular, $\int_M \partial \widehat{\psi} = 0$ and $\partial \psi_0$ is constant on each fibre. Therefore, $\int_M \partial \widehat{\psi} \cdot \partial \psi_0 \ dV_h = 0$, and it follows that

$$\int_{I \times M} \dot{g}(\nabla \widehat{\psi}, \nabla \psi_0) \ dV \ = \ 0. \tag{36}$$

The claimed (33) follows.

From (34) we also have that

$$\dot{g}(X, X) \leq 2 \cdot \max\{a, b\} \frac{\dot{\rho}}{\rho} \ g(X, X) \tag{37}$$

and hence combined with (33) we have

$$\int_{I \times M} \dot{g}(\nabla \psi, \nabla \psi) \ dV \ \leq \ 2 \cdot \max\{a, b\} \int_{I \times M} \frac{\dot{\rho}}{\rho} \ g(\nabla \widehat{\psi}, \nabla \widehat{\psi}) \ dV. \tag{38}$$

Substitution into (32) then yields

$$\begin{aligned} \dot{\lambda} \int_N \psi^2 \ dV \ &\geq \ -2 \max\{a, b\} \int_{I \times M} \frac{\dot{\rho}}{\rho} \cdot g(\nabla \widehat{\psi}, \nabla \widehat{\psi}) \ dV \\ &\quad + (a + bd) \int_{I \times M} \frac{\dot{\rho}}{\rho} \cdot \left(g(\nabla \psi, \nabla \psi) - \lambda \psi^2\right) dV \\ &\quad - C \cdot \lambda \int_K \psi^2 \ dV. \end{aligned} \tag{39}$$

Since $\dot{\rho}/\rho$ is homogeneous of degree -1, Lemma 3.4 applies to give

$$\begin{aligned} \left| \int_{I \times M} \frac{\dot{\rho}}{\rho} \cdot \left(g(\nabla \psi, \nabla \psi) - \lambda \psi^2\right) \ dV \right| \ &\leq \ C \int_{I \times M} \rho^{-2a-3} \cdot \psi^2 \ dV \\ &\quad + C'(\lambda + 1) \int_{(2I \setminus I) \times M} \psi^2 \ dV \end{aligned} \tag{40}$$

as well as the analogous estimate with ψ replaced by $\widehat{\psi}$. By combining these estimates with (39) and absorbing constants, we obtain the claim. $\square$

5. THE CASE $b \leq 0$

Theorem 5.1. *Let $a < -1$ and $b \leq 0$. Then each eigenvalue branch $\lambda(\epsilon)$ converges to a finite limit as ϵ tends to 0^+.*

Proof. We apply Theorem 4.1. Since $\dot{\rho}$ and ρ are positive and $\max\{a, b\} \leq 0$, the first term on the right hand side of (29) is nonnegative. Therefore, by Remark 3.5 we have

$$\text{(41)} \qquad \dot{\lambda} \; \geq \; -C\epsilon^{-2a-3} \; - \; C'(\lambda + 1)$$

for some constants C and C'. Since $\lambda \geq 0$, division of both sides by $\lambda + 1$ gives

$$\text{(42)} \qquad \frac{d}{d\epsilon} \log(\lambda + 1) \; \geq \; -C\epsilon^{-2a-3} - C'.$$

Since $a < -1$, the left hand side of (42) is integrable, and, moreover, the negative variation of $\log(\lambda + 1)$ over $]0, \epsilon[$ is $O(\epsilon^{-2a-2})$. Thus, since $-2a - 2 > 0$, the function $\log(\lambda + 1)$ has a limit as ϵ tends to 0^+. Thus, the claim follows via exponentiation. $\qquad\square$

Proposition 5.2. *Let $a \leq -1$ and $b < 0$. Then each eigenvalue branch $\lambda(\epsilon)$ remains bounded as ϵ tends to 0^+.*

Proof. Let $\delta = -\max\{a, b\} > 0$. Note that because $a \leq -1$, we have $-2a - 3 \geq -1$, and hence $\epsilon^{-2a-3} \leq \epsilon^{-1}$ for ϵ small. Thus, by using Theorem 4.1 and Remark 3.5, we obtain

$$\text{(43)} \qquad \frac{d}{d\epsilon}\lambda \; \geq \; (\delta\lambda - C) \cdot \epsilon^{-1} - C'.$$

If $\lambda > (C + 1)/\delta$, then the right hand side of (43) is positive for ϵ small. The claim follows. $\qquad\square$

6. AN A PRIORI ESTIMATE

Assumption 6.1. *In the sequel we will assume that $b > 0$ and either $a < -1$ or $a \leq -1$ and $a + bd = 0$.*

Theorem 6.2. *The quantity $\epsilon^{2b}\lambda(\epsilon)$ tends to a finite limit as ϵ tends to zero.*

Proof. By Remark 3.5 and Assumption 6.1, we have $\rho^{-2a-3} \leq C \cdot \epsilon^{-2a-3} \leq C\epsilon^{-1}$ for ϵ small. Thus, it follows from Theorem 4.1 and Proposition 3.7, that there exist positive constants C, C' such that

$$\text{(44)} \qquad \dot{\lambda} \; \geq \; -2b \cdot \lambda \cdot \epsilon^{-1} \; - \; C \cdot \epsilon^{-1} \; - \; C' \cdot \lambda \,.$$

Since $b > 0$, we have upon letting $c = C/2b$

$$\text{(45)} \qquad \dot{\lambda} \; \geq \; -2b \cdot (\lambda + c) \cdot \epsilon^{-1} \; - \; C \cdot \lambda.$$

Dividing by $\lambda + c$ gives

$$\text{(46)} \qquad \frac{d}{d\epsilon} \log(\lambda + c) \; \geq \; -2b \; \cdot \epsilon^{-1} \; - \; C.$$

Since $\frac{d}{d\epsilon}\log(\epsilon^{2b}) = 2b \cdot \epsilon^{-1}$, we obtain

$$(47) \qquad \frac{d}{d\epsilon}\log\left(\epsilon^{2b}(\lambda + c)\right) \geq -C.$$

It follows that the negative variation of $f(\epsilon) = \log\left(\epsilon^{2b}(\lambda + c)\right)$ over the interval $]0, \epsilon[$ is $O(\epsilon)$. It follows that $\lim_{\epsilon \to 0} f(\epsilon)$ is either finite, in which case $\lim_{\epsilon \to 0} \epsilon^{2b}\lambda(\epsilon)$ is finite, or $\lim_{\epsilon \to 0} f(\epsilon) = -\infty$ in which case $\lim_{\epsilon \to 0} \epsilon^{2b}\lambda(\epsilon) = 0$. In either case the limit exists. $\qquad\square$

7. A BOOTSTRAP

Let k_0 denote the infimum of all k such that the function $\epsilon^{2kb} \cdot \lambda(\epsilon)$ has a limit as ϵ tends to zero. By Theorem 6.2, we have $k_0 \leq 1$. The purpose of this section is to prove

Theorem 7.1. *If $k_0 < 1$, then $\lambda(\epsilon)$ tends to a finite limit as ϵ tends to zero.*

As a first step towards proving Theorem 7.1, we have the following

Proposition 7.2. *If there exist constants $C^* > 0$ and $k < 1$ such that*

$$(48) \qquad \int_{I \times M} \rho^{-1}\widehat{\psi}^2 \; dV \;\leq\; C^*\epsilon^{-k} \int_{I \times M} \widehat{\psi}^2 dV,$$

then $\lambda(\epsilon)$ tends to a finite limit as ϵ tends to zero.

Proof. Since $\dot{\rho}$ is homogeneous of degree zero, it is bounded. By Assumption 6.1, we have $-2a - 3 \leq -1$, and hence $\rho^{-2a-3} \leq \rho^{-1}$. Therefore, via Theorem 4.1 we find that

$$\dot{\lambda} \int_N \psi^2 \; dV \;\geq\; -C \cdot (\lambda + 1) \int_{I \times M} \rho^{-1} \cdot \widehat{\psi}^2 \; dV \;-\; C'(\lambda + 1) \int_K \psi^2 \; dV.$$

Hence by (48) and Remark 3.5

$$(49) \qquad \dot{\lambda} \;\geq\; -C'' \cdot (\lambda + 1) \cdot \epsilon^{-k}$$

for some positive C'' and ϵ small. Dividing by $\lambda + 1$ gives

$$(50) \qquad \frac{d}{d\epsilon}\log(\lambda + 1) \;\geq\; -C'' \cdot \epsilon^{-k}.$$

Since $k < 1$, the right hand side is integrable, and, in particular, the negative variation of $f(\epsilon) = \log(\lambda + c)$ is $O(\epsilon^{1-k})$. Therefore $\lim_{\epsilon \to 0} f(\epsilon)$ exists, and it follows that $\lambda(\epsilon)$ has a limit. $\qquad\square$

To verify (48)—and thus prove Theorem 7.1—we split the domain of integration of the integral on the left hand side according to whether Corollary 3.3 implies the convexity of $\|\psi\|_M^2$ or not. To be precise, let μ_1 denote the smallest non-zero eigenvalue of Δ_h. Define $A(\epsilon)$ to be set of t such that

$$(51) \qquad \lambda(\epsilon) \cdot \rho^{2b}(\epsilon, t) \;\leq\; \frac{\mu_1}{2}.$$

The key idea in what follows is that (51) and Corollary 3.3 imply that the function $t \to \|\psi\|_M^2(t)$ is convex enough to tame the singular behavior of $\frac{\dot{\rho}}{\rho}$ near $(0, 0)$. This heuristic will be made precise in Lemma 7.4.

In the following $B(\epsilon)$ will denote the set complement $I \setminus \frac{1}{2}A(\epsilon)$.

Proof of Theorem 7.1. We claim that if $k_1 > k_0$, then for sufficiently small ϵ

$$(52) \qquad \int_{B(\epsilon) \times M} \rho^{-1} |\widehat{\psi}|^2 \, dV \;\le\; \epsilon^{-k_1} \int_{I \times M} |\widehat{\psi}|^2 dV_g.$$

Indeed, from (51), we have $t \in B(\epsilon)$ if and only if

$$(53) \qquad \lambda(\epsilon) \cdot \rho^{2b}(\epsilon, 2t) \;\ge\; \frac{\mu_1}{2}.$$

If $k_1 > k_0$, then for sufficiently small ϵ, we have $\epsilon^{2bk_1} \cdot \lambda(\epsilon) \le \frac{\mu_1}{2}$, and hence for $t \in B(\epsilon)$

$$(54) \qquad \rho^{2b}(\epsilon, 2t) \;\ge\; \epsilon^{2bk_1}.$$

Therefore $\rho^{-1}(2\epsilon, 2t) \le (2\epsilon)^{-k_1}$ for $t \in B(\epsilon)$ and sufficiently small ϵ. The claimed (52) then follows from homogeneity and integration.

By applying Lemma 7.4 with $\mu = \mu_1$, $\delta = \frac{\mu}{2}$, and $\psi = \widehat{\psi}$, we obtain the complementary estimate. Indeed, since M is compact, the function $\widehat{\psi}$—defined by (28)—is orthogonal to the zero eigenspace of Δ_h. Hence for each $t \in I$

$$(55) \qquad \int_{\{t\} \times M} \widehat{\psi} \cdot \Delta_h \widehat{\psi} \, dV_h \;\ge\; \mu_1 \int_{\{t\} \times M} |\widehat{\psi}|^2 dV_h.$$

Thus, Lemma 7.4 applies to give

$$(56) \qquad \int_{\frac{1}{2}A(\epsilon) \times M} \rho^{-1} |\widehat{\psi}|^2 \, dV_g \;\le\; C \cdot \epsilon^{-(1-2b)} \int_{I \times M} |\widehat{\psi}|^2 dV_g.$$

Combining (52) and (56) gives us the desired (48) for all $k > \max\{1 - 2b, k_0\}$. $\qquad \square$

Remark 7.3. Without loss of generality, we may assume that $\rho < 1$ on $I \times I$.

Lemma 7.4. *Let $\mu > \delta > 0$ and let (ψ, λ) be a Laplace eigenbranch on $I \times M$. Let $A(\epsilon)$ denote the set of t that satsify*

$$(57) \qquad \lambda(\epsilon) \cdot \rho^{2b}(\epsilon, t) \;\le\; \mu - \delta.$$

Suppose that

$$(58) \qquad \lim_{\epsilon \to 0} \lambda(\epsilon) \cdot \rho^{2b}(\epsilon, 0) \;<\; \mu - \delta$$

and that for each t

$$(59) \qquad \int_{\{t\} \times M} \psi \cdot \Delta_h \psi \, dV_h \;\ge\; \mu \int_{\{t\} \times M} \psi^2 \, dV_h.$$

Then there exist $C > 0$ such that for small $\epsilon > 0$

$$(60) \qquad \int_{\frac{1}{2}A(\epsilon) \times M} \rho^{-1} \cdot \psi^2 \, dV_g \;\le\; C \cdot \epsilon^{2b-1} \int_{A(\epsilon) \times M} \psi^2 \, dV_g.$$

Proof. By Remark 7.3 we have $\delta\rho^{2b} \leq \delta$ and hence it follows from (57) that $\delta \leq \mu \cdot \rho^{-2b} - \lambda$. Therefore

$$
\begin{aligned}
\delta \int_{\frac{1}{2}A(\epsilon)\times M} \rho^{-1}\psi^2 \, dV &= \delta \int_{\frac{1}{2}A(\epsilon)} \rho^{-1}||\psi||_M^2 \cdot \rho^{a+bd} \, dt \\
&\leq \int_{\frac{1}{2}A(\epsilon)} (\mu\rho^{-2b} - \lambda) \cdot ||\psi||_M^2 \cdot \rho^{2b-1} \cdot \rho^{a+bd} dt. \\
&\leq \frac{1}{2} \int_{\frac{1}{2}A(\epsilon)} L(||\psi||_M^2) \cdot \rho^{2b-1} \cdot \rho^{a+bd} dt.
\end{aligned}
$$

(61)

Here the last inequality follows from (59) and Corollary 3.3.

We wish to apply integration by parts to the last integral in (61). Towards this end, let χ be a smooth function supported in $[-1, 1]$ with $\chi \equiv 1$ on $[-1/2, 1/2]$ and $\max|\chi| = 1$. Since ρ is positive, convex, and homogeneous of degree 1, the set $A(\epsilon)$ is a closed interval $[t_-(\epsilon), t_+(\epsilon)]$ that contains 0. For each $\epsilon > 0$ define

(62)
$$
\bar{\chi}(\epsilon, t) = \begin{cases} \chi\left(\frac{t}{t_+(\epsilon)}\right) & \text{for } t \geq 0 \\ \chi\left(\frac{t}{t_-(\epsilon)}\right) & \text{for } t \leq 0. \end{cases}
$$

Integration by parts shows that the operator L is symmetric on $L^2(I, \rho^{a+bd} \, dt)$ with Dirichlet boundary conditions. Thus,

(63)
$$
\int_{\frac{1}{2}A(\epsilon)} \bar{\chi} \cdot L(||\psi||_M^2) \cdot \rho^{2b-1} \cdot \rho^{a+bd} dt = \int_I ||\psi||_M^2 \cdot L(\bar{\chi} \cdot \rho^{2b-1}) \cdot \rho^{a+bd} \, dt.
$$

By (59) and Corollary 3.3, we have $L(||\psi||_M^2(t)) > 0$ for $t \in A(\epsilon)$. Thus, since $A(\epsilon) = \text{supp}(\bar{\chi})$, estimate (61) and (63) combine to give

(64)
$$
2\delta \int_{\frac{1}{2}A(\epsilon)\times M} \rho^{-1}\psi^2 \, dV \leq \int_{I\times M} L(\bar{\chi} \cdot \rho^{2b-1}) \cdot \psi^2 \, dV.
$$

Therefore, to verify (60), it will suffice to show that

(65)
$$
\left|L(\bar{\chi} \cdot \rho^{2b-1})\right| = O(\epsilon^{2b-1}).
$$

By homogeneity, the supremum of $\rho^c(\epsilon, t)$ over I is $O(\epsilon^c)$ for any constant c. We compute

(66)
$$
L(\bar{\chi}\rho^{2b-1}) = \bar{\chi} \cdot L(\rho^{2b-1}) + \beta \cdot \rho^{-2a+2b-2}\partial\rho \, \partial\bar{\chi} + \rho^{-2a+2b-1}\partial^2\bar{\chi}
$$

where $\beta = (-a + bd + 2c)$. The operator L adds $-2a - 2$ to the degree of a homogeneous function, and hence

(67)
$$
|L(\rho^{2b-1})| \leq \rho^{2a-3+2b} = O(\epsilon^{2a-3+2b})
$$

Since $a \geq -1$, we have $2a - 3 \geq -1$, and hence $|L(\rho^{2b-1})| = O(\epsilon^{2b-1})$.

To estimate the remaining two terms in (66), we need to estimate $|\partial\bar{\chi}|$ and $|\partial^2\bar{\chi}|$. To this end, consider $r(\epsilon) = \min\{|t_\pm(\epsilon)|\}$, the inner radius of $A(\epsilon)$. By Lemma 7.5, there exists $\eta > 0$ such that $|r(\epsilon)| \geq \eta \cdot \epsilon$ for all ϵ small. Therefore $|\partial\bar{\chi}| \leq r(\epsilon)^{-1} \cdot |\chi'| = O(\epsilon^{-1})$ and, similarly, $|\partial^2\bar{\chi}| = O(\epsilon^{-2})$.

The function $\rho^{-2a+2b-2}$ appearing in (66) is homogeneous of degree $-2a-2+2b \geq 2b$. Hence since $\partial\rho$ is homogeneous of degree zero,

(68)
$$
|\rho^{-2a+2b-2} \cdot \partial\rho \cdot \partial\bar{\chi}| = O(\epsilon^{2b-1}).
$$

A similar argument shows that

$$(69) \qquad |\rho^{-2a+2b-1} \cdot \partial\rho \cdot \partial^2 \bar{\chi}| = O(\epsilon^{2b-1})$$

The desired estimate (65) then follows from (67), (68), and (69). $\qquad\square$

Lemma 7.5. *Let (ψ, λ) be a Laplace eigenbranch on $I \times M$. Let $\mu > \delta > 0$ and let $A(\epsilon) = [t_-(\epsilon), t_+(\epsilon)]$ be defined as in (57). If (58) holds, then there exists $\eta > 0$ such that*

$$(70) \qquad |t_\pm(\epsilon)| \geq \eta \cdot \epsilon.$$

for small ϵ.

Proof. By definition, $t_\pm$ satisfies

$$\lambda(\epsilon) \cdot \rho^{2b}(\epsilon, t_\pm(\epsilon)) = \mu - \delta.$$

Thus, by homogeneity

$$(71) \qquad \lambda(\epsilon) \cdot \epsilon^{2b} \cdot \rho^{2b}(1, \epsilon^{-1} \cdot t_\pm(\epsilon)) = \mu - \delta.$$

By Theorem 6.2, $\lambda(\epsilon) \cdot \epsilon^{2b}$ tends to $c \leq 0$ as ϵ tends to zero. If $c = 0$, then since the right hand side of (71) is positive, $\epsilon^{-1} \cdot t_\pm(\epsilon)$ must tend to infinity as ϵ tends to zero.

By homogeneity, (58) implies that $c \cdot \rho^{2b}(1, 0) < \mu - \delta$. Thus if $c > 0$, then by (71) we have

$$\rho^{2b}(1, \epsilon^{-1} \cdot t_\pm(\epsilon)) = \frac{\mu - \delta}{c} > \rho^{2b}(1, 0)$$

By Remark 3.6 and the condition $b > 0$, the function $t \to \rho^{2b}(1, t)$ assumes a unique minimum at $t = 0$. Therefore, $\epsilon^{-1} \cdot t_\pm(\epsilon)$ is strictly bounded away from zero. The claim follows. $\qquad\square$

8. A second bootstrap

By homogeneity $\rho^{2b}(\epsilon, 0) = \rho^{2b}(1, 0) \cdot \epsilon^{2b}$, and hence, by Theorem 6.2, the limit

$$(72) \qquad \mu^* = \lim_{\epsilon \to 0} \rho^{2b}(\epsilon, 0) \cdot \lambda(\epsilon)$$

exists. The purpose of this section is to prove

Theorem 8.1. *If μ^* does not equal a positive eigenvalue of Δ_h, then $\lambda(\epsilon)$ tends to a finite limit as ϵ tends to zero.*

To prove Theorem 8.1, we will use the Δ_h spectral decomposition of ψ. To be precise, the orthogonal projection, $E_\mu : L^2(M, dV_h) \to L^2(M, dV_h)$, onto the μ-eigenspace of Δ_h extends fibre by fibre to an operator $\bar{E}_\mu : L^2(I \times M, dV_g) \to L^2(I \times M, dV_g)$. Set

$$(73) \qquad \psi^* = \bar{E}_{\mu*}(\psi)$$

$$\psi_+ \;=\; \sum_{\mu > \mu^*} \bar{E}_\mu(\psi)$$

$$\psi_- \;=\; \sum_{0 < \mu < \mu^*} \bar{E}_\mu(\psi).$$

Note that ψ^*, ψ_+ and ψ_- are all eigenfunctions of Δ_g with eigenvalue λ. The 0-eigenspace of Δ_h consists of the constant functions. Thus if μ^* is not a positive eigenvalue, then

(74) $$\psi = \psi_0 + \psi_- + \psi_+.$$

where ψ_0 is defined in (27).

Proof of Theorem 8.1. Since $a < 0$, by (34) we have

$$\dot{g}(\nabla\widehat{\psi}, \nabla\widehat{\psi}) \;\leq\; 2b \cdot \frac{\dot\rho}{\rho} \cdot \rho^{2b} \cdot h(\nabla_g\widehat{\psi}, \nabla_g\widehat{\psi})$$

Note that from (12) we obtain $h(\nabla_g\widehat{\psi}, \nabla_g\widehat{\psi}) = \rho^{-4b} h(\nabla_h\widehat{\psi}, \widehat{\nabla}_h\psi)$. Hence,

(75) $$\int_{I \times M} \dot{g}(\nabla\widehat{\psi}, \nabla\widehat{\psi}) \;\leq\; 2b \int_I \frac{\dot\rho}{\rho} \cdot \rho^{-2b} \left(\int_{\{t\} \times M} h(\nabla_h\widehat{\psi}, \nabla_h\widehat{\psi}) \; dV_h \right) dt.$$

By hypothesis we have $\psi^* = 0$ and hence $\widehat{\psi} = \psi_+ + \psi_-$. Applying Parseval's principle for Δ_h acting on $L^2(M, dV_h)$, we obtain

(76)
$$\int_{\{t\} \times M} h(\nabla_h\widehat{\psi}, \nabla_h\widehat{\psi}) \; dV_h \;=\; \int_{\{t\} \times M} h(\nabla_h\psi_-, \nabla_h\psi_-) \; dV_h \;+\; \int_{\{t\} \times M} h(\nabla_h\psi_+, \nabla_h\psi_+) \; dV_h$$

as well as

(77) $$\int_{\{t\} \times M} \psi^2 \; dV_h \;=\; \int_{\{t\} \times M} (\psi_-)^2 \; dV_h \;+\; \int_{\{t\} \times M} (\psi_+)^2 \; dV_h.$$

Thus, it follows from Lemmas 8.2 and 8.3 that there exists $k < 1$ such that

$$\int_{I \times M} \frac{\dot\rho}{\rho} \cdot \rho^{-2b} \cdot h(\nabla_h\widehat{\psi}, \nabla_h\widehat{\psi}) \; dV \;\leq\; \left(k \cdot \lambda + C\epsilon^{-2a-2} \right) \cdot \epsilon^{-1} \int_{I \times M} \widehat{\psi}^2 \; dV$$

(78) $$+ \; C' \cdot (\lambda + 1) \int_K \psi^2 \; dV.$$

Since $-2a - 2 \geq 0$, combining (78) with (75) and (33) yields $k < 1$ such that

(79)
$$\int_{I \times M} \dot{g}(\nabla\psi, \nabla\psi) \; dV \;\leq\; 2b \cdot k \cdot (\lambda + c) \cdot \epsilon^{-1} \int_{I \times M} \widehat{\psi}^2 \; dV \;+\; C' \cdot (\lambda + 1) \int_K \psi^2 dV$$

for some $c > 0$.

By substituting (79) into (32) and applying (40) to the $(a + bd)$-term in (32), one obtains

$$\dot\lambda \int_N \psi^2 \; dV \;\geq\; -2b \cdot k \cdot (\lambda + c) \cdot \epsilon^{-1} \int_{I \times M} \frac{\dot\rho}{\rho} \cdot \widehat{\psi}^2 \; dV$$

$$+ \; C'\epsilon^{-2a-3} \int_{I \times M} \psi^2 dV \;-\; C'' \cdot (\lambda + 1) \int_K \psi^2 \; dV.$$

for some constants C' and C''. Thus, since $a \leq -1$, there exists a positive constant c' such that

$$\dot{\lambda} \geq -2b \cdot k \cdot (\lambda + c') \cdot \epsilon^{-1} - C''(\lambda + c'),$$

Division by $\lambda + c'$ gives

$$\frac{d}{d\epsilon} \log(\lambda + c') \geq -2b \cdot k \cdot \epsilon^{-1} - C'',$$

and hence, by arguing as in the proof of Theorem 6.2, one finds that $\epsilon^{2kb} \cdot \lambda(\epsilon)$ converges as ϵ tends to zero. Therefore the claim follows from Theorem 7.1. $\square$

Lemma 8.2. *There exists $k < 1$ such that*

$$(80) \qquad \int_{I \times M} \frac{\dot{\rho}}{\rho} \cdot \rho^{-2b} h(\nabla_h \psi_-, \nabla_h \psi_-) \, dV \leq k \cdot \lambda \cdot \epsilon^{-1} \int_{I \times M} \psi_-^2 \, dV.$$

Proof. If μ^* is less than the smallest positive eigenvalue of Δ_h, then $\psi_- = 0$ and the claim follows. Otherwise, let μ_- be the largest Δ_h-eigenvalue that is less than μ^*. From the definition of ψ_- we have

$$(81) \qquad \int_{\{t\} \times M} h(\nabla_h \psi_-, \nabla_h \psi_-) \, dV_h \leq \mu_- \int_{\{t\} \times M} \psi_-^2 \, dV_h$$

Since, by hypothesis, $\frac{\mu_-}{\mu^*} < 1$, there exists $k < 1$ such that for sufficiently small ϵ

$$(82) \qquad \mu_- \leq k \cdot \rho^{2b}(\epsilon, 0) \cdot \lambda(\epsilon).$$

Hence, using the (global) hypothesis that $\rho^{2b}(\epsilon, 0) \leq \rho^{2b}(\epsilon, t)$ for all t, we have

$$(83) \qquad \rho^{-2b}(\epsilon, t) \cdot \mu_- \leq k \cdot \lambda(\epsilon).$$

Combining this with (81) gives

$$(84) \qquad \int_{I \times M} \frac{\dot{\rho}}{\rho} \rho^{-2b} h(\nabla_h \psi_-, \nabla_h \psi_-) \, dV_h \leq k \cdot \lambda \cdot \int_{I \times M} \frac{\dot{\rho}}{\rho} \psi_-^2 \, dV_h.$$

Therefore, the claim follows from Proposition 3.7. $\square$

Lemma 8.3. *There exists $k < 1$ such that*

$$\int_{I \times M} \frac{\dot{\rho}}{\rho} \cdot \rho^{-2b} h(\nabla_h \psi_+, \nabla_h \psi_+) \, dV \leq \left(k \cdot \lambda + C \epsilon^{-2a-2} \right) \cdot \epsilon^{-1} \int_{I \times M} \psi_+^2 \, dV$$

$$(85) \qquad\qquad\qquad\qquad + C' \cdot (\lambda + 1) \int_K \psi^2 \, dV.$$

Proof. By (12) and (1) we have

$$(86) \qquad \rho^{-2b} \cdot h(\nabla_h \psi_+, \nabla_h \psi_+) \leq \rho^{2b} \cdot h(\nabla_g \psi_+, \nabla_g \psi_+) \leq g(\nabla_g \psi_+, \nabla_g \psi_+).$$

Using Lemma 3.4, one obtains (40) with ψ replaced by ψ_+. From this and (86) it follows that the left hand side of (85) is bounded above by

$$(87) \quad \lambda \int_{I \times M} \frac{\dot{\rho}}{\rho} \cdot \psi_+^2 \, dV + C \cdot \epsilon^{-2a-3} \int_{I \times M} \psi_+^2 + C'(\lambda + 1) \int_{(2I \setminus I) \times M} \psi_+^2 \, dV.$$

Thus it will suffice to show that

$$(88) \qquad \int_{I \times M} \frac{\dot{\rho}}{\rho} \, \psi_+^2 \, dV \;\leq\; k \cdot \epsilon^{-1} \int_{I \times M} \psi_+^2 \, dV$$

for some $k < 1$.

Towards verifying (88), we will apply Lemma 7.4. Namely, let μ_+ to be the smallest of all eigenvalues that are greater than μ^* and let $\delta = (\mu_+ - \mu^*)/2$. Note that from the definition of ψ_+, for each $t \in I$

$$\int_{\{t\} \times M} \psi_+ \cdot \Delta_h \psi_+ \, dV_h \;\geq\; \mu_+ \int_{\{t\} \times M} \psi_+^2 \, dV_h.$$

Note also that (58) follows in this case from (72). Therefore Lemma 7.4 provides $C > 0$ such that

$$(89) \qquad \int_{\frac{1}{2} A(\epsilon) \times M} \rho^{-1} \cdot \psi_+^2 \, dV_g \;\leq\; C \cdot \epsilon^{2b-1} \int_{A(\epsilon) \times M} \psi_+^2 \, dV_g.$$

By Lemma 7.5, there exists $\eta > 0$ such that $[-\eta\epsilon, \eta\epsilon] \subset A(\epsilon)$. Define

$$k' = \sup_{2|s| < \eta} \frac{\dot{\rho}(1, s)}{\rho(1, s)}$$

It follows from Proposition 3.7 that $k' < 1$. By homogeneity, we have

$$(90) \qquad \frac{\dot{\rho}(\epsilon, t)}{\rho(\epsilon, t)} \;\leq\; k' \cdot \epsilon^{-1}$$

for all $|t| < 2^{-1} \eta \cdot \epsilon$. In particular, estimate (90) holds for all $t \in B(\epsilon)$. Thus, by integrating this estimate over $B(\epsilon)$ and combining with (89) we obtain

$$(91) \qquad \int_{I \times M} \frac{\dot{\rho}}{\rho} \, \psi_+^2 \, dV \;\leq\; (k' + C\epsilon^{2b}) \cdot \epsilon^{-1} \int_{I \times M} \psi_+^2 \, dV$$

Therefore, since $b > 0$, the claimed (88) is proven. $\qquad\qquad\qquad\square$

9. A vacuous case

Theorem 9.1. *Each eigenvalue branch converges to a finite limit.*

Proof. We assume that the hypothesis of Theorem 8.1 is not satisfied and derive a contradiction. Namely, we assume that μ^* is a positive Δ_h-eigenvalue and obtain a contradiction in the form of two conflicting estimates. In particular, it is enough to show that there exist constants $C_1, C_2 > 0$ such that for small ϵ

$$(92) \qquad C_1 \cdot \epsilon^{-b-a-1} \;\leq\; \lambda(\epsilon) - \mu^* \cdot \rho^{-2b}(\epsilon, 0) \;\leq\; C_2 \cdot \epsilon^{-2a-2}.$$

This is impossible since $b > 0$ and $-a - 1 \leq 0$ and hence $-b - a - 1 < -2a - 2$. The respective sides of (92) are given below as Lemma 9.2 and Lemma 9.3. $\quad\square$

Lemma 9.2 (Left Hand Estimate). *Suppose that μ^* is a positive Δ_h-eigenvalue. Then there exists a constant $C > 0$ such that for all small ϵ*

$$(93) \qquad C \cdot \epsilon^{-b-a-1} \;\leq\; \lambda - \mu^* \cdot \rho^{-2b}(\epsilon, 0).$$

Proof. We first claim that it suffices to show that there exists $\delta > 0$ such that for small ϵ

$$(94) \qquad \dot{\lambda} \ \geq \ -2b \cdot \lambda \cdot \epsilon^{-1} \cdot \left(1 - \delta \cdot \epsilon^{-a-1} \lambda^{-\frac{1}{2}}\right).$$

Indeed, since $\rho^{2b}(\epsilon, 0) \cdot \lambda(\epsilon) \to \mu^*$, there exists $\delta' > 0$ such that $\lambda^{\frac{1}{2}} \geq \delta' \epsilon^{-b}$ for small ϵ. Hence we would have

$$(95) \qquad \dot{\lambda} \ \geq \ -2b\lambda \cdot \epsilon^{-1} - \delta \cdot \delta' \cdot \epsilon^{-b-a-2}.$$

Note that since $(\dot{\rho}/\rho)(\epsilon, 0) = \epsilon^{-1}$

$$(96) \qquad \frac{d}{d\epsilon}\left(\rho^{2b}(\epsilon, 0) \cdot \lambda(\epsilon)\right) \ \geq \ \rho^{2b}(\epsilon, 0) \cdot \left(2b \cdot \lambda \cdot \epsilon^{-1} + \lambda'(\epsilon)\right).$$

Thus, since $\rho^{2b}(\epsilon, 0) = \rho(1, 0) \cdot \epsilon^{2b}$ it would follow from (95) that there exists a constant $\delta'' > 0$ such that

$$(97) \qquad \frac{d}{d\epsilon}\left(\rho^{2b}(\epsilon, 0) \cdot \lambda(\epsilon)\right) \ \geq \ \delta'' \cdot \epsilon^{b-a-2}$$

Note that since $a \leq -1$ and $b > 0$, we have $b - a - 2 > -1$. Thus, since $\lim_{\epsilon \to 0} \rho^{2b}(\epsilon, 0) \cdot \lambda(\epsilon) = \mu^*$, we could then integrate (97) over $[0, \epsilon]$ and would find that

$$(98) \qquad \rho^{2b}(\epsilon, 0) \cdot \lambda(\epsilon) \ - \mu^* \ \geq \ \delta'' \cdot \epsilon^{b-a-1}.$$

Since $\rho^{2b}(\epsilon, 0) = \rho^{2b}(1, 0) \cdot \epsilon^{2b}$, we would then obtain (93) by dividing both sides of (98) by $\rho^{2b}(\epsilon, 0)$.

Recall from (73) that ψ^* denotes the fibrewise projection of ψ onto the μ^* eigenspace of Δ_h. Letting $\bar{\psi} = \psi_+ + \psi_-$, we have $\hat{\psi} = \psi^* + \overline{\psi}$. We claim that to verify (94) it suffices to show that

$$(99) \qquad \int_{(\epsilon I) \times M} \frac{\dot{\rho}}{\rho} \cdot (\psi^*)^2 \ dV \ \leq \ r(\epsilon) \cdot \epsilon^{-1} \cdot \int_{(\epsilon I) \times M} (\psi^*)^2 \ dV$$

where

$$(100) \qquad r(\epsilon) = 1 - \delta' \cdot \epsilon^{-a-1} \lambda^{-\frac{1}{2}}$$

and $\delta' > 0$. To see this, note that by Proposition 3.7, there exists $k < 1$ such that $(\dot{\rho}/\rho)(\epsilon, t) \leq k \cdot \epsilon^{-1}$ for $|t| \geq \epsilon$. It follows that

$$(101) \qquad \int_{(I \setminus \epsilon I) \times M} \frac{\dot{\rho}}{\rho} \cdot (\psi^*)^2 \ dV \ \leq \ k \cdot \epsilon^{-1} \cdot \int_{(I \setminus \epsilon I) \times M} (\psi^*)^2 \ dV.$$

Since $\mu^* > 0$, we have $\lambda \to \infty$, and hence since $a + 1 \leq 0$, for any $k < 1$, we have $r(\epsilon) > k$ for all sufficiently small ϵ. Thus, by combining (99) and (101), we would obtain

$$(102) \qquad \int_{I \times M} \frac{\dot{\rho}}{\rho} \cdot (\psi^*)^2 \ dV \ \leq \ r(\epsilon) \cdot \epsilon^{-1} \cdot \int_{I \times M} (\psi^*)^2 \ dV$$

for ϵ small. Applying the argument in (86) and (87) with ψ^+ replaced by ψ^*, we would have

$$\int_{I \times M} \frac{\dot{\rho}}{\rho} \cdot \rho^{-2b} \cdot h(\nabla_h \psi^*, \nabla_h \psi^*) dV \ \leq \ \left(\lambda \cdot r(\epsilon) + C\epsilon^{-2a-2}\right) \cdot \epsilon^{-1} \cdot \int_{I \times M} (\psi^*)^2 \ dV$$

$$(103) \qquad\qquad\qquad\qquad\qquad\qquad\qquad + C' \cdot (\lambda + 1) \int_K \psi^2 dV$$

(In this and what follows, C and C' represent generic constants.) Recall that in (78) we had $\widehat{\psi} = \psi_+ + \psi_-$, and hence we have (78) with $\widehat{\psi}$ replaced by $\overline{\psi}$:

$$\int_{I \times M} \frac{\dot{\rho}}{\rho} \cdot h(\nabla_h \overline{\psi}, \nabla_h \overline{\psi}) dV \quad \leq \quad (\lambda \cdot k + C \cdot \epsilon^{-2a-2}) \cdot \epsilon^{-1} \cdot \int_{I \times M} (\overline{\psi})^2 \, dV$$

$$(104) \qquad \qquad \qquad + C' \cdot (\lambda + 1) \int_K \psi^2 dV.$$

As pointed out above, $r(\epsilon) > k$ for small ϵ. Hence by applying Parseval's principle as in (76) and (77) to $\widehat{\psi} = \psi^* + \overline{\psi}$, we could combine (103) and (104) to find that

$$\int_{I \times M} \frac{\dot{\rho}}{\rho} \cdot \rho^{-2b} \cdot h(\nabla_h \widehat{\psi}, \nabla_h \widehat{\psi}) \, dV \quad \leq \quad (\lambda \cdot r(\epsilon) + C \cdot \epsilon^{-2a-2}) \cdot \epsilon^{-1} \int_{I \times M} \widehat{\psi}^2 \, dV$$

$$(105) \qquad \qquad \qquad + C' \cdot (\lambda + 1) \int_K \psi^2 \, dV.$$

Combining this with (75) would yield $k < 1$ such that

$$(106)$$
$$\int_{I \times M} \dot{g}(\nabla \widehat{\psi}, \nabla \widehat{\psi}) \, dV \leq \left(2b \cdot \lambda \cdot r(\epsilon) + C \cdot \epsilon^{-2a-2}\right) \cdot \epsilon^{-1} \int_{I \times M} \widehat{\psi}^2 \, dV + C' \cdot (\lambda + 1) \int_K \psi^2 dV.$$

where C denotes a (generic) positive constant. By substituting (106) into (32) and applying (40) to the $(a + bd)$-term in (32) and using (33), one would obtain

$$(107)$$
$$\dot{\lambda} \int_N \psi^2 \, dV \geq -\left(2b \cdot \lambda \cdot r(\epsilon) + C\epsilon^{-2a-2}\right) \cdot \epsilon^{-1} \int_{I \times M} \psi^2 dV - C'(\lambda + 1) \int_K \psi^2 \, dV.$$

Note that $C'(\lambda + 1) < 2b\lambda \cdot r \cdot \epsilon^{-1}$ for small ϵ. Hence from (107) one would have

$$(108) \qquad \dot{\lambda} \int_N \psi^2 \, dV \geq -\left(2b \cdot \lambda \cdot r(\epsilon) + C\epsilon^{-2a-2}\right) \cdot \epsilon^{-1} \int_N \psi^2 dV$$

for small ϵ. Since $\lambda \to \infty$ and $-a - 1 \geq 0$

$$(109) \qquad C \cdot \epsilon^{-2a-2} \leq \frac{\delta'}{2} \cdot \epsilon^{-a-1}$$

for small ϵ. Thus by choosing $\delta = \delta'/2$ and recalling (100) we would obtain (94) from (108). And hence (94) follows from (99).

As a first step toward the verification of (99), we rescale in ϵ. In particular, let $h(s) = \dot{\rho}\rho^{-1}(1, s)$, and $\sigma(s) = \rho^{2a}(1, s)$, and for each $\epsilon > 0$ and $m \in M$, define

$$(110) \qquad v_{\epsilon, m}(s) = \rho^{\frac{-a+bd}{2}}(1, s) \cdot \psi^*(\epsilon s, m).$$

Then using homogeneity, we find that

$$(111) \qquad \int_{(\epsilon I) \times M} \frac{\dot{\rho}}{\rho} \cdot (\psi^*)^2 \, dV_g = \epsilon^{a+bd} \int_M \int_I h(s) \cdot v_{\epsilon, m}(s)^2 \cdot \sigma(s) \, ds \, dV_h$$

and

$$(112) \qquad \int_{(\epsilon I) \times M} (\psi^*)^2 \, dV_g = \epsilon^{a+bd+1} \int_M \int_I v_{\epsilon, m}(s)^2 \cdot \sigma(s) \, ds \, dV_h.$$

Since $\Delta\psi^* = \lambda\psi^*$ and since $\psi^*(t,\cdot)$ belongs to the μ^* eigenspace of Δ_h, we have $-L\psi^* + \mu^*\rho^{-2b}\psi^* = \lambda\psi^*$ from (10). It follows from (110) and homogeneity that $v_{\epsilon,m}$ satifies the following ordinary differential equation

$$(113) \qquad v''(s) = \eta \cdot \rho^{2a-2b}(1,s) \cdot \left(\frac{\mu^*}{\lambda \cdot \epsilon^{2b}} - \rho^{2b}(1,s)\right) \cdot v(s) + g(s) \cdot v(s)$$

where $\eta = \epsilon^{2a+2} \cdot \lambda(\epsilon)$ and $g(s)$ is a bounded smooth function. Hence, by (111) and (112), to prove (99) it will suffice to prove that there exists $\delta > 0$ such that for any solution v to (113) we have

$$(114) \qquad \int_I h(s) \cdot v(s)^2 \cdot \sigma(s) \, ds \leq \left(1 - \delta \cdot \eta^{-\frac{1}{2}}\right) \int_I v(s)^2 \cdot \sigma(s) \, ds.$$

Towards verification of (114) we apply Lemma A.1 to (113). Indeed, by hypothesis $\mu^* \cdot \lambda^{-1} \cdot \epsilon^{-2b}$ is bounded, and hence Lemma A.1 applies to give a constant $C > 0$ such that

$$(115) \qquad \int_{\eta^{-\frac{1}{4}}I} v(s)^2 \cdot \sigma(s) \, ds \leq C \int_{\eta^{-\frac{1}{4}}(2I\backslash I)} v(s)^2 \cdot \sigma(s) \, ds.$$

It follows that

$$(116) \qquad \int_I v(s)^2 \cdot \sigma(s) \, ds \leq (C+1) \int_{I\backslash(2\eta^{-\frac{1}{4}}I)} v(s)^2 \cdot \sigma(s) \, ds.$$

By Proposition 3.7 and the strict convexity of ρ, there exists $\delta' > 0$ such that $h(s) = \dot\rho\rho^{-1}(1,s) \leq 1 - \delta' \cdot s^2$ for $s \in I$. It follows that

$$\int_I h(s) \cdot v(s)^2 \cdot \sigma(s) \, ds \leq \int_I v(s)^2 \cdot \sigma(s) \, ds - 4\delta' \cdot \eta^{-\frac{1}{2}} \int_{I\backslash(2\eta^{-\frac{1}{4}}I)} v(s)^2 \cdot \sigma(s) \, ds.$$

By substituting (116) and choosing $\delta = 4\delta' \cdot (1+C)^{-1}$ we obtain (114). The proof is complete. $\square$

Lemma 9.3 (Right Hand Estimate). *Suppose that μ^* is a positive Δ_h-eigenvalue. Then there exists a constant C such that for small $\epsilon > 0$*

$$(117) \qquad f(\epsilon) = \lambda(\epsilon) - \mu^* \cdot \rho^{-2b}(\epsilon, 0) \leq C\epsilon^{-2a-2}.$$

Proof. It will suffice to show that

$$(118) \qquad \frac{d}{d\epsilon} f(\epsilon) \geq -(2a \cdot f(\epsilon) + C\epsilon^{-2a-2}) \cdot \epsilon^{-1}.$$

Indeed, we may suppose that $2 \cdot f(\epsilon) \geq C\epsilon^{-2a-2}$, for otherwise we are done. Hence (118) implies

$$\frac{d}{d\epsilon} f(\epsilon) \geq -(2a+2) \cdot f(\epsilon) \cdot \epsilon^{-1}.$$

By Lemma 9.2, we have $f > 0$ for small ϵ, and thus division would give

$$\frac{d}{d\epsilon} \log\left(\epsilon^{2a+2} \cdot f(\epsilon)\right) \geq 0.$$

By integrating over $[\epsilon, \delta]$ with δ small we would find that

$$\log\left(\frac{\delta^{2a+2} \cdot f(\delta)}{\epsilon^{2a+2} \cdot f(\epsilon)}\right) \geq 0.$$

Exponentiation would then give (117) with $C = \delta^{2a+2} \cdot f(\delta)$.

To verify (118) we will estimate $\lambda'(\epsilon)$ using the methods of §4. From (34) and (12) we have

$$(119) \qquad \dot{g}(\nabla\widehat{\psi}, \nabla\widehat{\psi}) \;=\; 2a \cdot \frac{\dot{\rho}}{\rho} \cdot \rho^{-2a} \cdot (\partial_t \widehat{\psi})^2 \;+\; 2b \cdot \frac{\dot{\rho}}{\rho} \cdot \rho^{-2b} \cdot h(\nabla_h \widehat{\psi}, \nabla_h \widehat{\psi}).$$

By substituting (17) into (40) one obtains that

$$\left| \iint_{I\times M} \frac{\dot{\rho}}{\rho} \cdot \rho^{-2a} (\partial_t \widehat{\psi})^2 dV + \int_{I\times M} \frac{\dot{\rho}}{\rho} \cdot \rho^{-2b} \cdot h(\nabla_h \widehat{\psi}, \nabla_h \widehat{\psi}) dV - \lambda \int_{I\times M} \frac{\dot{\rho}}{\rho} \widehat{\psi}^2 dV \right|$$

$$(120) \qquad\qquad\qquad \le C \int_{I\times M} \rho^{-2a-3} \cdot \widehat{\psi}^2 \; dV \; + C'(\lambda+1) \int_K \widehat{\psi}^2 \; dV$$

Thus, by integrating (119) and using (120) we find that

$$\int_{I\times M} \dot{g}(\nabla\widehat{\psi}, \nabla\widehat{\psi}) dV \;\; \le \;\; 2a \cdot \lambda \int_{I\times M} \frac{\dot{\rho}}{\rho} \, \widehat{\psi}^2 \; dV$$

$$(121) \qquad\qquad\qquad + (2b - 2a) \cdot \int_{I\times M} \frac{\dot{\rho}}{\rho} \, \rho^{-2b} \cdot h(\nabla_h \widehat{\psi}, \nabla_h \widehat{\psi}) \; dV$$

$$\qquad\qquad\qquad + C \cdot \epsilon^{-2a-3} \int_{I\times M} \widehat{\psi}^2 \; dV + C'(\lambda+1) \int_{(2I\setminus I)\times M} \widehat{\psi}^2 \; dV.$$

We may estimate the righthand side of (121) by splitting the integral over the sum $\widehat{\psi} = \psi^* + \overline{\psi}$ where $\overline{\psi} = \psi_+ + \psi_-$. To be precise, apply Parseval's principle—as in (76)—to find that

$$\int_{\{t\}\times M} h(\nabla_h \widehat{\psi}, \nabla_h \widehat{\psi}) \; dV_h \;=\; \int_{\{t\}\times M} h(\nabla_h \psi^*, \nabla_h \psi^*) \; dV_h \;+\; \int_{\{t\}\times M} h(\nabla_h \overline{\psi}, \nabla_h \overline{\psi}) \; dV_h$$

as well as

$$(122) \qquad \int_{\{t\}\times M} (\widehat{\psi})^2 \; dV_h \;=\; \int_{\{t\}\times M} (\psi^*)^2 \; dV_h \;+\; \int_{\{t\}\times M} (\overline{\psi})^2 \; dV_h.$$

It follows that if we let $E(\widehat{\psi})$ denote the right hand side of (121), then

$$(123) \qquad\qquad\qquad E(\widehat{\psi}) \;=\; E(\psi^*) \;+\; E(\overline{\psi}).$$

By definition $\Delta_h \psi^* = \mu^* \psi^*$ and hence $\int_M h(\nabla_h \psi^*, \nabla_h \psi^*) dV_h = \mu^* \int_M (\psi^*)^2 dV_h$. It follows that

$$(124) \quad E(\psi^*) \;\le\; \big(q(\epsilon) + C\epsilon^{-2a-2}\big) \cdot \epsilon^{-1} \int_{I\times M} (\psi^*)^2 \; dV + C'(\lambda+1) \int_K \psi^2 \; dV.$$

where

$$(125) \qquad\qquad q(\epsilon) = 2a \cdot \lambda(\epsilon) + (2b - 2a) \cdot \mu^* \cdot \rho^{-2b}(\epsilon, 0).$$

To estimate $E(\overline{\psi})$, apply (120) 'in reverse' to find that

$$E(\overline{\psi}) \;\; \le \;\; 2a \int_{I\times M} \frac{\dot{\rho}}{\rho} \cdot \rho^{-2a} \cdot (\partial_t \overline{\psi})^2 \; dV \;+\; 2b \int_{I\times M} \frac{\dot{\rho}}{\rho} \cdot \rho^{-2b} \cdot h(\nabla_h \overline{\psi}, \nabla_h \overline{\psi}) \; dV$$

$$+ C'' \cdot \epsilon^{-2a-3} \int_{I\times M} \overline{\psi}^2 \; dV + C'''(\lambda+1) \int_{(2I\setminus I)\times M} \overline{\psi}^2 \; dV.$$

Thus by using the assumption $a < 0$, the formula (76), and Lemmas 8.3 and 8.2, we obtain $k < 1$ such that

$$(126) \quad E(\overline{\psi}) \ \leq \ (2k \cdot b\lambda + C'' \cdot \epsilon^{-2a-2}) \cdot \epsilon^{-1} \int_{I \times M} \overline{\psi}^2 \, dV + C'''(\lambda + 1) \int_K \overline{\psi}^2 \, dV$$

By combining (33), (121), (123), (124), and (126), we find (generic) constants C, C' such that

$$\int_{I \times M} \dot{g}(\nabla \psi, \nabla \psi) \, dV \ \leq \ (q + C\epsilon^{-2a-2})\epsilon^{-1} \int_{I \times M} (\psi^*)^2 \, dV$$
$$+ (2kb\lambda + C\epsilon^{-2a-2})\epsilon^{-1} \int_{I \times M} (\overline{\psi})^2 \, dV \ + \ C'(\lambda + 1) \int_K \psi^2 \, dV.$$

Hence by substituting this into (32) and applying (40) to the middle term in (32), we obtain

$$\dot{\lambda} \int_N \psi^2 \, dV \ \geq \ -(q + C\epsilon^{-2a-2})\epsilon^{-1} \int_{I \times M} (\psi^*)^2 \, dV$$
$$- (2kb\lambda + C\epsilon^{-2a-2})\epsilon^{-1} \int_{I \times M} (\overline{\psi})^2 \, dV \ - \ C'(\lambda + 1) \int_K \psi^2 \, dV$$

for possibly different (generic) constants. By using (122), we find that

$$\dot{\lambda} \int_N \psi^2 \, dV \ \geq \ - \big(\max\{q, 2kb\lambda\} + C\epsilon^{-2a-2}\big) \epsilon^{-1} \int_{I \times M} \widehat{\psi}^2 \, dV$$
$$(127) \qquad\qquad - C'(\lambda + 1) \int_K \psi^2 \, dV.$$

We claim that there exists $\epsilon_0 > 0$ such that for all $\epsilon \leq \epsilon_0$

$$(128) \qquad\qquad q(\epsilon) \ \geq \ 2b \cdot k \cdot \lambda(\epsilon).$$

for all small ϵ. To see this, first note that we cannot have $q < 2b \cdot k \cdot \lambda$ for all small ϵ. For then by (127) we would have that $\dot{\lambda} \geq -(2bk\lambda + C) \cdot \epsilon^{-1} - C'\lambda$ for all small ϵ. Using the argument of Theorem 6.2, we would then find that $\epsilon^{2kb}\lambda$ is bounded and hence λ would converge by Theorem 7.1. This would contradict the assumption that $\mu^* > 0$.

Hence there exists $\epsilon_0 > 0$ such that (128) is true for $\epsilon = \epsilon_0$. A calculation shows that (128) holds if and only if

$$(129) \qquad\qquad \left(\frac{1-k}{-a+b}\right) \ \geq \ \frac{f(\epsilon)}{\lambda(\epsilon)}.$$

Thus to prove (128) for $\epsilon \leq \epsilon_0$, it suffices to show that f/λ is increasing for small ϵ. To see that this is true, note that by homogeneity $\partial \rho^{-2b}(\epsilon, 0) = -2b \cdot \rho^{-2b}(\epsilon, 0) \cdot \epsilon^{-1}$, and hence

$$(130) \qquad\qquad f'(\epsilon) = \lambda'(\epsilon) + 2b\mu^* \cdot \rho^{-2b}(\epsilon, 0) \cdot \epsilon^{-1}.$$

By Lemma 9.2, $f(\epsilon) > 0$ for small ϵ. Thus from (130) we have $\partial \log(f(\epsilon)) \geq \partial \log(\lambda(\epsilon))$, and hence $\partial(f/\lambda) \geq 0$ as desired.

Therefore, since (128) holds true, (127) yields

$$(131)$$
$$\dot{\lambda} \int_N \psi^2 \, dV \ \geq \ - \big(q(\epsilon) + C \cdot \epsilon^{-2a-2}\big) \cdot \epsilon^{-1} \int_{I \times M} \widehat{\psi}^2 \, dV \ - \ C'(\lambda + 1) \int_K \psi^2 \, dV.$$

Moreover, for small ϵ we have $2k'b \cdot \epsilon^{-1} \geq 2C'$, and hence by (127) we have $q > C'(\lambda + 1)$. it follows from (131) that

$$\dot{\lambda} \geq -\left(q(\epsilon) + C \cdot \epsilon^{-2a-2}\right) \cdot \epsilon^{-1}.$$

Into this substitute (125), and use both (130) and the definition of f in (117) to obtain (118) as desired. $\qquad\square$

Appendix A. On the width of scars

Let $f > 0$, g and h, be continuous functions on $\mathbb{R}$. Let I be an interval containing 0, and let $\beta \in \mathbb{R}$ and $\eta \in \mathbb{R}^+$. Consider the ordinary differntial equation

$$(132) \qquad w''(s) = \eta \cdot f(s) \cdot \left(\beta + s^2 \cdot h(s)\right) \cdot w(s) + g(s) \cdot w(s).$$

Lemma A.1. *Let $\sigma > 0$ be a positive continuous function on $\mathbb{R}$. There exists a constant $C > 0$ such that for any solution w to (132)*

$$(133) \qquad \int_{\lambda^{-\frac{1}{4}} I} w^2(s)\ \sigma(s)\ ds \ \leq\ C \int_{\lambda^{-\frac{1}{4}}(2I \setminus I)} w^2(s)\ \sigma(s)\ ds.$$

Proof. Let $x(u) = w(\eta^{-\frac{1}{4}} u)$. Then (133) is equivalent to

$$(134) \qquad \int_I x^2(u)\ \sigma(\eta^{-\frac{1}{4}} u)\ du \ \leq\ C \int_{2I \setminus I} x^2(u)\ \sigma(\eta^{-\frac{1}{4}} u)\ du.$$

From (132), we see that x satisfies

$$(135) \qquad x''(u) = \eta^{\frac{1}{2}} \cdot \beta \cdot f(\eta^{-\frac{1}{4}} u) \cdot x(u) + \left(u^2 \cdot h(\eta^{-\frac{1}{4}} u) + g(\eta^{-\frac{1}{4}} u)\right) \cdot x(u).$$

If $\eta^{\frac{1}{2}} \cdot \beta$ is large and positive, then $x(u)$ is uniformly convex on I, and (134) follows. If $\eta^{\frac{1}{2}} \cdot \beta$ is large and negative, then $x(u)$ oscillates rapidly. In particular, by [CrnHlb] Chapter V §4, x differs from a solution to $y''(u) + (\eta^{\frac{1}{2}}\beta) \cdot y(u)$ in C^0-norm by order $(\eta^{\frac{1}{2}}\beta)^{-\frac{1}{2}}$. A straightforward calculation shows that (134) holds uniformly for y and hence w for $\eta^{\frac{1}{2}}\beta$ positive and sufficiently large. For $\eta^{\frac{1}{2}}\beta$ bounded, the claim follows from continuous dependence on parameters, the linearity of the equation, and the fact that the L^2-norm cannot vanish on any nontrivial interval. $\qquad\square$

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